Brownian motion and Stochastic Calculus
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Assignment 5—solutions

Exercise 1

Let B be a standard Brownian motion. For each $n \geq 1$, let $B^{(n)}$ be the (random) function such that $B^{(n)}_t = B_t$ for all $t \in 2^{-n}\mathbb{N}_0$ and such that $B^{(n)}$ is linear on the intervals $[i2^{-n}, (i+1)2^{-n}]$ for all $i \geq 0$ (these processes appeared in the dyadic construction of Brownian motion). Fix $\varepsilon > 0$ and let $f \colon [0,1] \longrightarrow \mathbb{R}$ be continuous with $f(0) = 0$.

1) Show that $\mathbb{P}\big[\sup_{t\in[0,1]} |B_t - B^{(n)}_t| \leq \varepsilon/3\big] \longrightarrow 1$, as $n \to \infty$.

2) Prove that for all $n \in \mathbb{N}$, $B^{(n)}$ and $B - B^{(n)}$ are $\mathbb{P}$-independent.

3) Using uniform continuity of f, establish that $\mathbb{P}\big[\sup_{t\in[0,1]} |B_t - f(t)| \leq \varepsilon\big] > 0$.

1) **Since B is continuous, it is uniformly continuous on $[0,1]$ and thus**

$$\sup_{[0,1]} |B - B^{(n)}| \leq \sup_{s,t\in[0,1]\colon\, |s-t|\leq 2^{-n}} |B_s - B_t| \longrightarrow 0, \text{ as } n \to \infty.$$

Since almost sure convergence implies convergence in probability, the result follows.

2) **Since $B^{(n)}$ and $B - B^{(n)}$ are both continuous jointly centred Gaussian processes, it suffices to show that**

$$C(s,t) := \mathbb{E}^{\mathbb{P}}\big[B^{(n)}_s (B_t - B^{(n)}_t)\big] = 0, \text{ for all } s,t \geq 0.$$

Let us write $s = 2^{-n}(k+\alpha)$ and $t = 2^{-n}(l+\beta)$ for $k,l \geq 0$ and $\alpha,\beta \in [0,1)$. Then

$$B^{(n)}_s = (1-\alpha)B_{2^{-n}k} + \alpha B_{2^{-n}(k+1)}, \text{ and } B_t - B^{(n)}_t = (1-\beta)(B_{2^{-n}(l+\beta)} - B_{2^{-n}l}) + \beta(B_{2^{-n}(l+\beta)} - B_{2^{-n}(l+1)}).$$

So we can compute the covariances as follows

$$C(s,t) = \begin{cases} (1-\alpha)(2^{-n}k - 2^{-n}k) + \alpha(2^{-n}(k+1) - 2^{-n}(k+1)) = 0, \ k < l, \\ (1-\beta)(2^{-n}(l+\beta) - 2^{-n}l) + \beta(2^{-n}(l+\beta) - 2^{-n}(l+1)) = 0, \ k \geq l, \end{cases}$$

as required. One can prove this result alternatively by noting that the $B^{(n)}$ appear in the construction of Brownian motion (at the n'th iteration) and from there it is immediate that $B - B^{(n)}$ is independent of $B^{(n)}$.

3) **Let $f^{(n)} \colon [0,1] \to \mathbb{R}$ be the function which agrees with f on $[0,1] \cap 2^{-n}\mathbb{N}_0$ and is linear on the intervals $[i2^{-n}, (i+1)2^{-n}]$ for all $0 \leq i \leq 2^n - 1$. By the triangle inequality**

$$\begin{aligned} \sup_{[0,1]} |B - f| &\leq \sup_{[0,1]} |B - B^{(n)}| + \sup_{[0,1]} |f - f^{(n)}| + \sup_{[0,1]} |f^{(n)} - B^{(n)}| \\ &\leq \sup_{[0,1]} |B - B^{(n)}| + \sup_{s,t\in[0,1]\colon\, |s-t|\leq 2^{-n}} |f_s - f_t| + \sup_{i=1,\dots,2^n} |f_{i2^{-n}} - B_{i2^{-n}}|. \end{aligned}$$

By uniform continuity of f, we can take n sufficiently large such that the second term above is $\leq \epsilon/3$. Thus

$$\begin{aligned} \mathbb{P}\Big[\sup_{t\in[0,1]} |B_t - f(t)| \leq \varepsilon\Big] &\geq \mathbb{P}\bigg[\sup_{t\in[0,1]} |B_t - B^{(n)}_t| \leq \varepsilon/3, \sup_{i\in\{1,\dots,2^n\}} |f_{i2^{-n}} - B_{i2^{-n}}| \leq \varepsilon/3\bigg] \\ &= \mathbb{P}\bigg[\sup_{t\in[0,1]} |B_t - B^{(n)}_t| \leq \varepsilon/3\bigg]\mathbb{P}\bigg[\sup_{i\in\{1,\dots,2^n\}} |f_{i2^{-n}} - B_{i2^{-n}}| \leq \varepsilon/3\bigg], \end{aligned}$$

where we used the independence from 2). The first term of the product tends to 1 as $n \to \infty$ by 1) and the second term is > 0 since $(B_{i2^{-n}} : i \in \{1, \ldots, 2^n\})$ is a non-degenerate Gaussian vector. This completes the proof.

Exercise 2

Let B be a standard Brownian motion. We will now show that $\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,1]} |B_t|^p\right] < +\infty$ for all $p < +\infty$

1) Show that

$$\sup_{t\in[0,1]} |B_t| \le \sum_{n=1}^{+\infty} \sup_{i\in\{0,\ldots,2^n-1\}} \left|B_{(i+1)2^{-n}} - B_{i2^{-n}}\right|.$$

2) For $p \ge 1$, deduce that

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,1]} |B_t|^p\right]^{1/p} \le \sum_{n=1}^{+\infty} \left(\sum_{i=0}^{2^n-1} \mathbb{E}^{\mathbb{P}}\left[|B_{(i+1)2^{-n}} - B_{i2^{-n}}|^p\right]\right)^{1/p}.$$

3) Hence deduce that $\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,1]} |B_t|^p\right] < +\infty$ for all $p < +\infty$ sufficiently large and therefore actually for all $p \in (0, \infty)$.

1) **Any $t \in [0,1) \cap 2^{-m}\mathbb{N}$ can be written as $t = b_1 2^{-1} + \cdots + b_m 2^{-m}$ for some $b_i \in \{0,1\}$ and therefore**

$$|B_t| = |B_t - B_0| \le \sum_{k=1}^{m} |B_{b_1 2^{-1}+\cdots+b_k 2^{-k}} - B_{b_1 2^{-1}+\cdots+b_{k-1} 2^{-(k-1)}}| \le \sum_{k\ge1} \sup_{i\in\{0,\ldots,2^k-1\}} |B_{(i+1)2^{-k}} - B_{i2^{-k}}|.$$

Since $\cup_{m\in\mathbb{N}}[0,1) \cap 2^{-m}\mathbb{N}$ is dense in $[0,1]$, the statement follows.

2) **By Minkowski's inequality and by bounding the supremum by a sum, we get**

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,1]} |B_t|^p\right]^{1/p} &\le \sum_{n\in\mathbb{N}^\star} \left(\mathbb{E}^{\mathbb{P}}\left[\sup_{i\in\{0,\ldots,2^n-1\}} |B_{(i+1)2^{-n}} - B_{i2^{-n}}|^p\right]\right)^{1/p} \\ &\le \sum_{n\in\mathbb{N}^\star} \left(\sum_{i=0}^{2^n-1} \mathbb{E}^{\mathbb{P}}\left[|B_{(i+1)2^{-n}} - B_{i2^{-n}}|^p\right]\right)^{1/p}.\end{aligned}$$

Let $N \sim N(0,1)$, then since $B_{(i+1)2^{-n}} - B_{i2^{-n}}$ and $2^{-n/2}N$ have the same law, we get $\mathbb{E}^{\mathbb{P}}[|B_{(i+1)2^{-n}} - B_{i2^{-n}}|^p] = 2^{-np/2}\mathbb{E}^{\mathbb{P}}[|N|^p]$ and hence

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,1]} |B_t|^p\right]^{1/p} \le \mathbb{E}^{\mathbb{P}}[|N|^p]^{1/p} \sum_{n\in\mathbb{N}^\star} 2^{n(1/p-1/2)} < +\infty,$$

for $p > 2$. Since for $0 < p' < p$, $\sup_{t\in[0,1]} |B_t|^{p'} \le 1 + \sup_{t\in[0,1]} |B_t|^p$, the general case follows immediately.

Exercise 3

For a compact set $K \subset \mathbb{R}$, we define its lower Minkowski content of dimension $d > 0$ to be

$$m_d(K) = \liminf_{n\to\infty} \frac{1}{n^d} \sum_{i\in\mathbb{Z}} \mathbf{1}_{\{K\cap[i/n,(i+1)/n]\neq\emptyset\}} \in [0,\infty].$$

Let B be a standard Brownian motion and define $K := \{t \in [0,1] \colon B_t = 0\}$. The goal of this question is to show that for $d > 1/2$, $m_d(K) = 0$, $\mathbb{P}$–a.s. (which means that the lower Minkowski dimension of K is $\le 1/2$, $\mathbb{P}$–a.s.).

1) Show that $m_d(K)$ is measurable.

2) Prove that

$$\mathbb{E}^{\mathbb{P}}[m_d(K)] \leq \liminf_{n\to\infty} \frac{1}{n^d} \sum_{i=0}^{n-1} \mathbb{P}\big[K \cap [i/n,(i+1)/n] \neq \emptyset\big] \leq \liminf_{n\to\infty} \frac{1}{n^d} \sum_{i=0}^{n-1} \mathbb{P}\bigg[\sup_{t\in[0,1/n]} \big|B_{i/n+t} - B_{i/n}\big| \geq |B_{i/n}|\bigg].$$

3) Using the scaling and the weak Markov property of Brownian motion, show that

$$\mathbb{P}\bigg[\sup_{t\in[0,1/n]} \big|B_{i/n+t} - B_{i/n}\big| \geq |B_{i/n}|\bigg] = \mathbb{P}\bigg[\sup_{t\in[0,1]} |B_t| \geq \sqrt{i}|N|\bigg],$$

where $N \sim N(0,1)$ is independent of B.

4) Using the previous exercise and 3) above, show that for all $\alpha \in (0,1/2)$ there exists $c'_\alpha > 0$ such that whenever $i \in \mathbb{N}^\star$, we have

$$\mathbb{P}\bigg[\sup_{t\in[0,1/n]} \big|B_{i/n+t} - B_{i/n}\big| \geq |B_{i/n}|\bigg] \leq c'_\alpha/i^\alpha.$$

1. Deduce that $\mathbb{E}^{\mathbb{P}}[m_d(K)] = 0$ and hence $m_d(K) = 0$, $\mathbb{P}$–a.s. for $d > 1/2$.

1) **It suffices to observe that** $\inf_{t\in[i/n,(i+1)/n]} |B_t| = \inf_{t\in[i/n,(i+1)/n]\cap\mathbb{Q}} |B_t|$ **is measurable, and hence so is**

$$\mathbf{1}_{\{K\cap[i/n,(i+1)/n]\neq\emptyset\}} = \mathbf{1}_{\{\inf_{t\in[i/n,(i+1)/n]} |B_t|=0\}}\ .$$

The result follows since linear combinations, infima and limits preserve measurability.

2) **By Fatou's lemma and the linearity of the expectation, we obtain the first inequality. For the second inequality, we observe that** $K \cap [i/n,(i+1)/n] \neq \emptyset$ **if and only if** $B_t = 0$ **for some** $t \in [i/n,(i+1)/n]$ **and so necessarily** $\sup_{t\in[0,1/n]} |B_{i/n+t} - B_{i/n}| \geq |B_{i/n}|$ **implying**

$$\mathbb{P}\big[K \cap [i/n,(i+1)/n] \neq \emptyset\big] \leq \mathbb{P}\bigg[\sup_{t\in[0,1/n]} |B_{i/n+t} - B_{i/n}| \geq |B_{i/n}|\bigg].$$

3) **We first observe that**

$$\mathbb{P}\bigg[\sup_{t\in[0,1/n]} |B_{i/n+t} - B_{i/n}| \geq |B_{i/n}|\bigg] = \mathbb{P}\bigg[\sup_{t\in[0,1]} |\sqrt{n}B_{(i+t)/n} - \sqrt{n}B_{i/n}| \geq |\sqrt{n}B_{i/n}|\bigg].$$

Since $(\sqrt{n}B_{t/n} \colon t \geq 0)$ **and** B **have the same law, we deduce**

$$\mathbb{P}\bigg[\sup_{t\in[0,1]} |\sqrt{n}B_{(i+t)/n} - \sqrt{n}B_{i/n}| \geq |\sqrt{n}B_{i/n}|\bigg] = \mathbb{P}\bigg[\sup_{t\in[0,1]} |B_{i+t} - B_i| \geq |B_i|\bigg].$$

By the weak Markov property, $B_{i+\cdot} - B_i$ **and** B_i **are** $\mathbb{P}$**-independent,** $B_{i+\cdot} - B_i$ **and** B **have the same law, and** $\sqrt{i}N$ **and** B_i **have the same law. All of this implies that**

$$\mathbb{P}\bigg[\sup_{t\in[0,1]} |B_{i+t} - B_i| \geq |B_i|\bigg] = \mathbb{P}\bigg[\sup_{t\in[0,1]} |B| \geq \sqrt{i}|N|\bigg].$$

4) **By a union bound, we obtain**

$$\mathbb{P}\bigg[\sup_{t\in[0,1]} |B_t| \geq \sqrt{i}|N|\bigg] \leq \mathbb{P}\big[|N| \leq i^{-\alpha}\big] + \mathbb{P}\bigg[\sup_{t\in[0,1]} |B_t| \geq i^{1/2-\alpha}\bigg] \leq 2(2\pi)^{-1/2} i^{-\alpha} + i^{p(\alpha-1/2)} \mathbb{E}^{\mathbb{P}}\bigg[\sup_{t\in[0,1]} |B_t|^p\bigg],$$

whenever $p > 0$. By taking $p > 0$ such that $p(1/2 - \alpha) > \alpha$, we obtain the claim making use the previous exercise.

5) **By** 2) **and** 4)**, for** $\alpha \in (0, 1/2)$ **and** $d > 1/2$

$$\mathbb{E}^{\mathbb{P}}[m_d(K)] \leq c'_\alpha \liminf_{n\to\infty} \frac{1}{n^d}\left(1 + \sum_{i=1}^{n-1} i^{-\alpha}\right) \leq c''_\alpha \cdot \liminf_{n\to\infty} n^{1-\alpha-d},$$

for some constant $c''_\alpha > 0$. By choosing α sufficiently close to $1/2$, we get $1 - \alpha - d < 0$ which completes the proof.

Exercise 4

A function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is called locally Hölder continuous of order α at $x \in D$ if there exists $\delta > 0$ and $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|^\alpha$ for all $y \in D$ with $|x - y| \leq \delta$. A function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is called locally Hölder continuous of order α, if it is locally Hölder continuous of order α at each $x \in D$.

1) Let $Z \sim N(0, 1)$. Prove that $\mathbb{P}[|Z| \leq \varepsilon] \leq \varepsilon$ for any $\varepsilon \geq 0$.

2) Prove that for any $\alpha > \frac{1}{2}$, $\mathbb{P}$–almost all Brownian paths are nowhere on $[0, 1]$ locally Hölder-continuous of order α.

   Hint: take any $M \in \mathbb{N}$ satisfying $M(\alpha - \frac{1}{2}) > 1$ and show that the set $\{W_\cdot(\omega)$ is locally α-Hölder at some $s \in [0, 1]\}$ is contained in the set

$$B := \bigcup_{C\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \bigcap_{n=m}^{+\infty} \bigcup_{k=0}^{n-M} \bigcap_{j=1}^{M} \left\{ \left|W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)\right| \leq \frac{C}{n^\alpha} \right\}.$$

3) The *Kolmogorov–Čentsov theorem* states that an $\mathbb{R}$-valued process X on $[0, T]$ satisfying

$$\mathbb{E}^{\mathbb{P}}\left[|X_t - X_s|^\gamma\right] \leq C|t - s|^{1+\beta},\ (s, t) \in [0, T]^2,$$

   where γ, β, and C are positive, has a $\mathbb{P}$-modification which is locally Hölder-continuous of order α for all $\alpha < \beta/\gamma$. Use this to deduce that Brownian motion has for every $\alpha < 1/2$ a version which is locally Hölder-continuous of order α.

1) **The density $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ of Z is bounded by $\frac{1}{\sqrt{2\pi}} \leq \frac{1}{2}$. So**

$$\mathbb{P}[|Z| \leq \varepsilon] = \mathbb{P}[-\varepsilon \leq Z \leq \varepsilon] = \int_{-\varepsilon}^{\varepsilon} f(x)\mathrm{d}x \leq \frac{1}{2}2\varepsilon = \varepsilon.$$

2) **Take any $\alpha > \frac{1}{2}$ and let $M \in \mathbb{N}$ satisfying $M(\alpha - \frac{1}{2}) > 1$. If $W_\cdot(\omega)$ is locally Hölder-continuous of order α at the point $s \in [0, 1]$, there exists a constant C_h so that $|W_t(\omega) - W_s(\omega)| \leq C_h|t - s|^\alpha$ for t near s. Then $|W_{\frac{k}{n}}(\omega) - W_{\frac{k-1}{n}}(\omega)| \leq \mathit{const} \cdot n^{-\alpha}$ for all large enough n, for $\frac{k}{n}$ near s and M successive indices k. The set $\{W_\cdot(\omega)$ is locally α-Hölder at some $s \in [0, 1]\}$ is therefore contained in**

$$B := \bigcup_{C\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} \bigcup_{k\in\{0,\ldots,n-M\}} \bigcap_{j=1}^{M} \left\{ |W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)| \leq \frac{C}{n^\alpha} \right\}.$$

We show that this is a $\mathbb{P}$–null set. As the above Brownian increments are i.i.d. and distributed as $N(0, \frac{1}{n})$, we have, with $Z \sim N(0, 1)$, since $\mathbb{P}[|Z| \leq \varepsilon] \leq \varepsilon$ for any $\varepsilon \geq 0$ (see 1)**), that**

$$\mathbb{P}\left[\bigcap_{j=1}^{M} \{|W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)| \leq C\frac{1}{n^\alpha}\}\right] = \left(\mathbb{P}\left[|Z| \leq \frac{C}{n^{\alpha-1/2}}\right]\right)^M \leq C^M n^{-M(\alpha-\frac{1}{2})}. \tag{0.1}$$

Now, we have for any $n \geq m$

$$D_m := \bigcap_{n\geq m} \bigcup_{k=0}^{n-M} \bigcap_{j=1}^{M} \left\{ |W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)| \leq \frac{C}{n^\alpha} \right\} \subseteq \bigcup_{k=0}^{n-M} \bigcap_{j=1}^{M} \left\{ |W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)| \leq \frac{C}{n^\alpha} \right\},$$

and therefore, due to (0.1), since $M(\alpha - \frac{1}{2}) > 1$, we get

$$\mathbb{P}[D_m] \leq \limsup_{n\to\infty} \mathbb{P}\left[\bigcup_{k=0}^{n-M} \bigcap_{j=1}^{M} \left\{ |W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)| \leq \frac{C}{n^\alpha} \right\} \right] \leq \limsup_{n\to\infty} \left\{ nC^M n^{-M(\alpha-\frac{1}{2})} \right\} = 0.$$

Therefore, being a countable union of $\mathbb{P}$–null sets, B is such that $\mathbb{P}[B] = 0$.

3) **Let $Y_\sigma \sim \mathcal{N}(0, \sigma^2)$ for any $\sigma \geq 0$. We note that $\mathbb{E}^{\mathbb{P}}[Y_\sigma^m] = C_m \sigma^m$, where $C_m = \mathbb{E}^{\mathbb{P}}[Y_1^m]$. Thus**

$$\mathbb{E}^{\mathbb{P}}\big[|W_t - W_s|^{2n}\big] = C_{2n}|t-s|^n, \text{ for all } n \in \mathbb{N}.$$

Writing $\gamma_n := 2n$ and $\beta_n := n-1$ yields that

$$\mathbb{E}^{\mathbb{P}}\big[|W_t - W_s|^{\gamma_n}\big] = C_{2n}|t-s|^{1+\beta_n}, \text{ for all } n \in \mathbb{N}.$$

Now, fix $\alpha < \frac{1}{2}$. As $\frac{\beta_n}{\gamma_n} < \frac{1}{2}$ for any $n \in \mathbb{N}$ and $\frac{\beta_n}{\gamma_n}$ converges to $\frac{1}{2}$, we find big enough N such that $\alpha < \frac{\beta_N}{\gamma_N}$. Thus, we get the result applying the Kolmogorov–Čentsov theorem